On the Asymptotic Behavior of the Coefficient Field of Newforms Modulo p

Master's Thesis Presentation

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Background

Quick recap of modular forms

Definition
The *coefficient field* of a modular form f is the subfield of $\mathbb{C}$ generated by all the coefficients a_n of its q-expansion. That is $\mathbb{Q}_f := \mathbb{Q}(a_n(f) | n \in \mathbb{N})$.

Normalized eigenform

Definition
A Modular form that is an eigenvector for T_n where $n \in \mathbb{N}$ is called an *eigenform*. Additionally, an eigenform is said to be *normalized* if the q-coefficient in its Fourier series is one, i.e.

$$f = a_0 + q + \sum_{i=2}^{\infty} a_i q^i.$$

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$$\{\text{Normalized eigenforms in } M_k(N;\mathbb{C})\} \leftrightarrow Hom_{\mathbb{C}-algebra}(\mathbb{T}_{\mathbb{C}}(M_k(N,\mathbb{C})),\mathbb{C}).$$

$$M_k(N;\mathbb{C}) \times \mathbb{T}_{\mathbb{C}}(M_k(N;\mathbb{C})) \to \mathbb{C}, \quad (f,T) \to a_1(Tf).$$

Eichler-Shimura Corollary

Corollary
Let $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_k(\Gamma_1(N);\mathbb{C})$ *be a normalized Hecke eigenform.*

Then $\mathbb{Q}_f := \mathbb{Q}(a_n(f)|n \in \mathbb{N})$ *is a number field of degree less than or equal to* $\dim_{\mathbb{C}}(S_k(\Gamma_1(N,\mathbb{C}))$.

Modular Symbols

$$\mathbb{T}_R(M_k) \simeq \mathbb{T}_R(\mathscr{M}_k)$$

Newforms

$$M_k(N;\mathbb{C}) = M_k(N;\mathbb{C})^{eis} \oplus S_k(N;\mathbb{C})$$
$$S_k(N;\mathbb{C}) = S_k(N;\mathbb{C})^{old} \oplus S_k(N;\mathbb{C})^{new}.$$

Elementary Methods

Maeda's Conjecture

Conjecture (Maeda)
For any k and any normalized eigenform $f \in S_k(1)$, the coefficient field $\mathbb{Q}_f$ has degree equal to $d_k := \dim_{\mathbb{C}} S_k(1;\mathbb{C})$ and the Galois group of its normal closure over $\mathbb{Q}$ is the symmetric group S_{d_k}.

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A consequence: Characteristic polynomial of T_2 on $S_k(1)$ is irreducible for any k.

$$\mathbb{Q}_f := \mathbb{Q}(a_2(f))$$

Dedekind–Kummer

Theorem (Dedekind–Kummer)
Let F be a number field and $\alpha \in \mathcal{O}_F$ be such that $F = \mathbb{Q}(\alpha)$. Let $f \in \mathbb{Z}[x]$ be the minimal polynomial of α over $\mathbb{Z}$ and suppose p does not divide $ind(\alpha)$. Let

$$\overline{f} = \prod_{j=1}^{k} \overline{f_j}^{e_j} \in \mathbb{F}_p[x]$$

be the factorization in monic irreducible polynomials, and define $P_j := (p, f_j(\alpha))$ where f_j is any lift of $\overline{f}_j$ to $\mathbb{Z}[x]$. Then

$$p\mathcal{O}_F = \prod_{j=1}^{k} P_j^{e_j}.$$

Algorithms

Algorithm 1

Data: $k \geq 2$, $N \geq 2$
Result: Residue degrees of the Hecke Algebra of $S_k(N, \mathbb{C})$ over $\mathbb{F}_p$

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while $\dim(M) \neq d$ **do**
    $M \leftarrow M + \langle T_n \rangle$;
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end
Get a $\mathbb{F}_p$ basis of M : $T^{(1)}, \ldots, T^{(d)}$;
Localize $M = \prod_{\mathfrak{p}|p} M_{\mathfrak{p}}$;
Calculate the residue degrees of $M_{\mathfrak{p}}$;

Algorithm 2

Data: A basis $\{T^{(1)}, \ldots, T^{(d)}\}$ of the commutative finite dimensional $\mathbb{F}_p$-algebra $\mathbb{T}$

Result: Local components of the Hecke algebra

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        Find minimal polynomial P of T on S_i;

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    for *irreducible factor p_i in the factorization of P* **do**

      $S_{new} \leftarrow S_{new} + [Ker(p_i^{e_i}(T))]$

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        end

    end

    $S \leftarrow S_{new}$;

end

return S

Calculating the Residue Degree

$$\mathbb{F}_p^n = \bigoplus Ker(p_i(T^{(1)})^{e_i}) \supseteq \bigoplus Ker(p_i(T^{(1)})). \quad S_i := Ker(p_i(T_1)^{e_i}).$$

$$S_i = \bigoplus Ker(p_{ij}(T^{(2)})^{e_{ij}}). \qquad M \cdot S_i \subseteq S_i.$$

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$$S_i = \bigoplus Ker(p_{ij}(T^{(2)})^{e_{ij}}). \qquad M \cdot S_i \subseteq S_i.$$

$$\mathbb{T}_{\mathbb{F}_p,\mathfrak{p}} = \cap_g \ker(g^e) \implies d = \mathrm{lcm}(deg(g))$$

Heuristic

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Question: Is the maximal residue degree, a_p, of primes above p in $\mathbb{Q}_f$ related to b_n, the average maximum length of a cycle in a permutation of $\mathcal{S}_n$?

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$$\lambda := \lim_{n\to\infty} \frac{b_n}{n} \approx 0.6243\ldots \text{ (Golomb and Gaal)}$$

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$$\lambda := \lim_{n\to\infty} \frac{b_n}{n} \approx 0.6243\ldots \text{ (Golomb and Gaal)}$$

$$\lim_{n\to\infty} a_p(N)/dim(S_k(N;\mathbb{C})) \sim \lambda/2$$

Symmetric Group and Frobenius

Proposition
Let M/K be a separable field extension of degree d, where K is a number field. Let $\mathfrak{p}$ be a prime of K and $\mathfrak{P}$ be a prime of L dividing $\mathfrak{p}$. We suppose that $\mathfrak{P}/\mathfrak{p}$ is unramified. Then the cycle lengths in the cycle decomposition of $Frob_{\mathfrak{P}/p} \in S_d$ are precisely the residue degrees of the primes of M lying above $\mathfrak{p}$.

Chebotarev

Let f be a monic irreducible polynomial of deg d with integer coefficients with a root α. Let $K = \mathbb{Q}(\alpha)$. Let A denote the set of unramified primes, P be a partition of $d = (d_1, \ldots, d_n)$ and A_P denote the set of primes such that f factors over p as

$$f = f_1 \ldots f_n$$

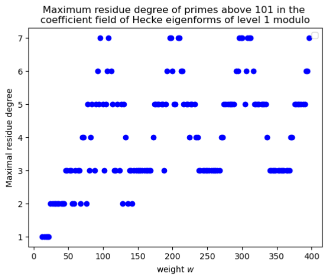
where f_i is an irreducible polynomial of degree d_i.

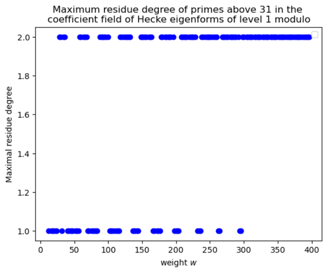
Now consider the Galois group $G \subseteq S_d$ of the number field K. Let G_P be the set of elements of G consisting of cycles of length $d_1, \ldots, d_n$. Then

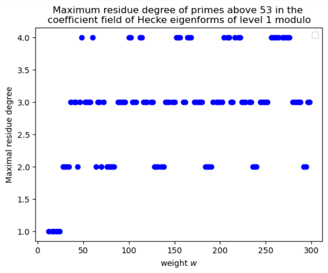
$$\delta(A_P) = \lim_{N\to\infty} \frac{\#\{p \in A_P : p \leq N\}}{\#\{p \in A : p \leq N\}} = \frac{G_P}{G}.$$

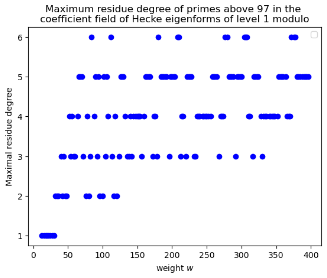
Experiments

Plots: Fixed Level $N = 1$





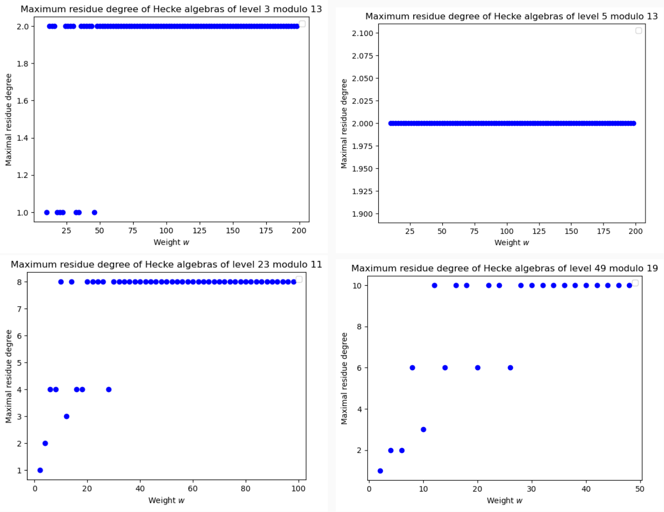




Jochnowitz

If we know all eigenforms of level 1 and weight $\leq p+1$, then we essentially get all the eigenforms over $\mathbb{F}_p$ in all weights by multiplying those of low weights by A_p, where $A_p = 1$ is a modular form of weight $p-1$ and level 1 over $\mathbb{F}_p$.

Fixed Level: $N \geq 1$



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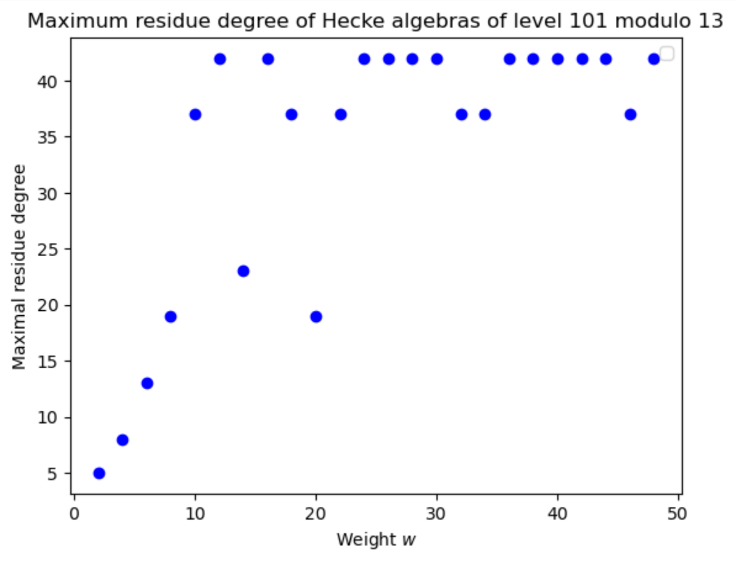

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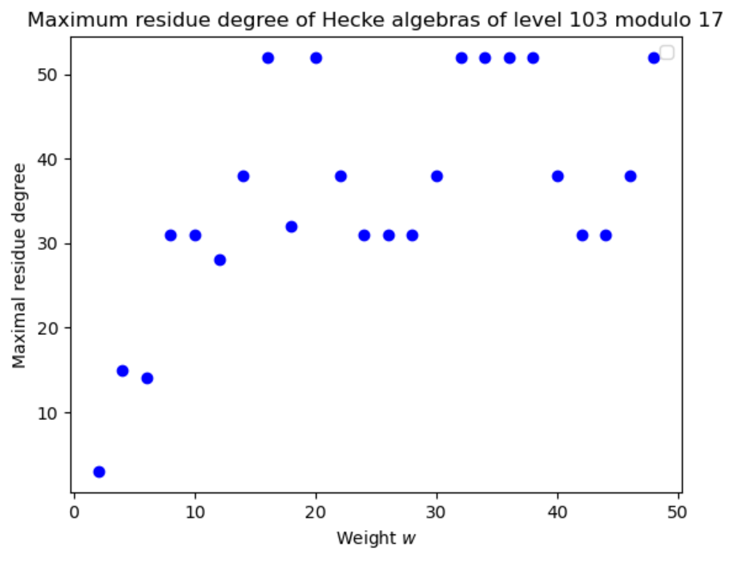
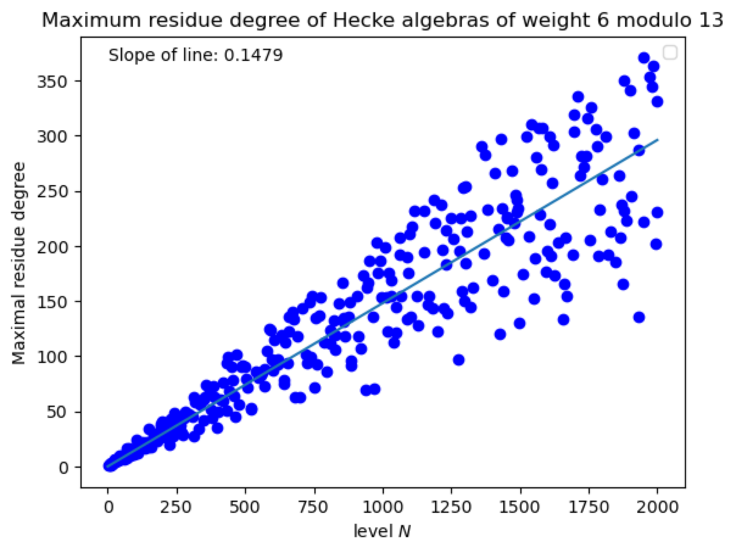
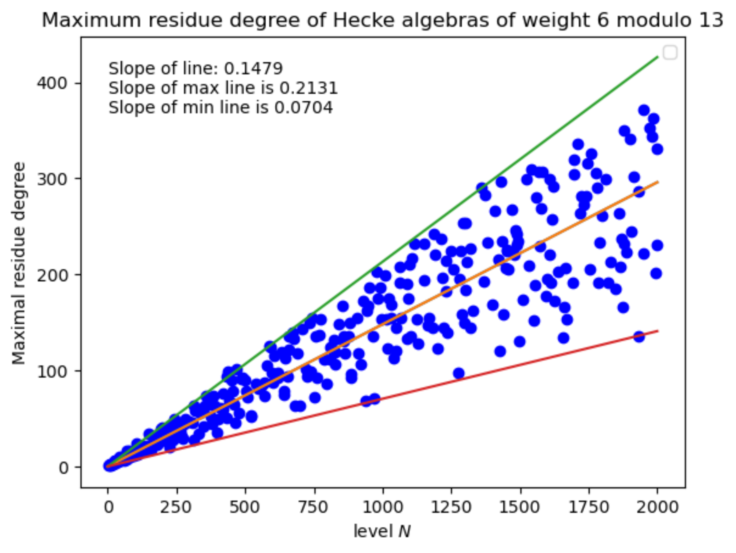


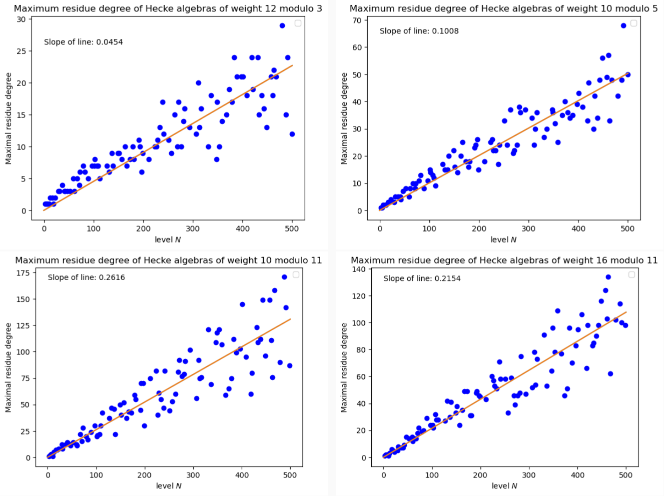
Fixed Weight



Fixed Weight



Fixed Weight



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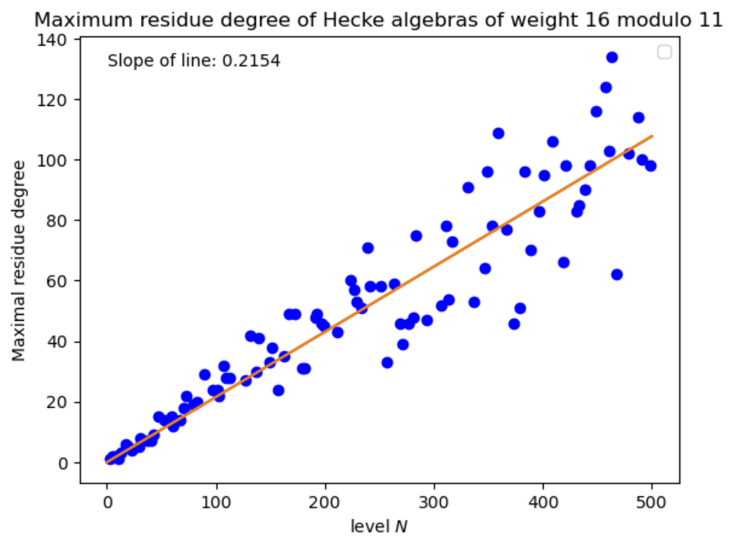
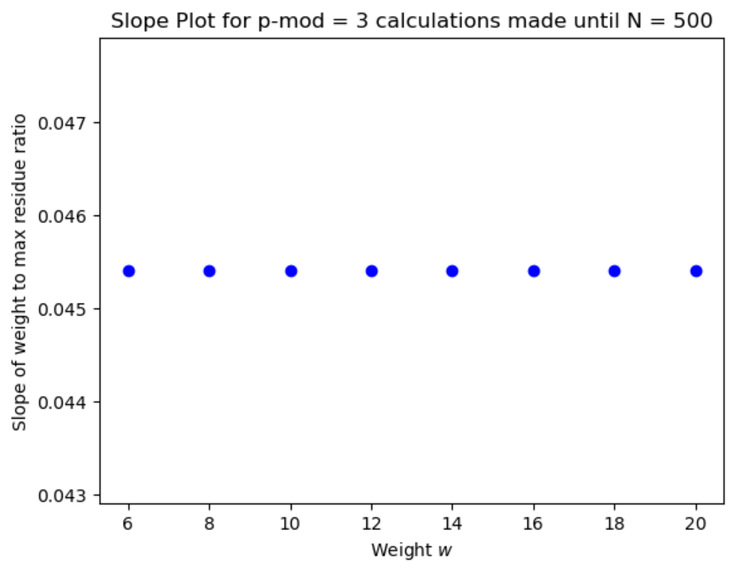
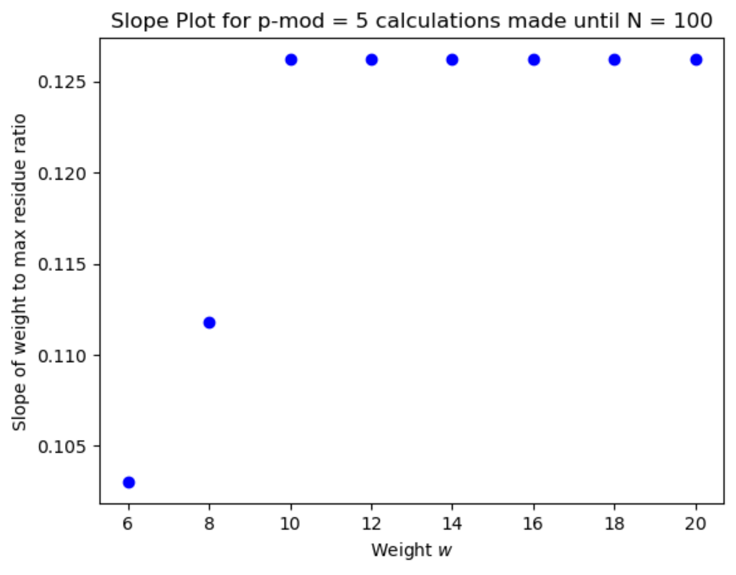


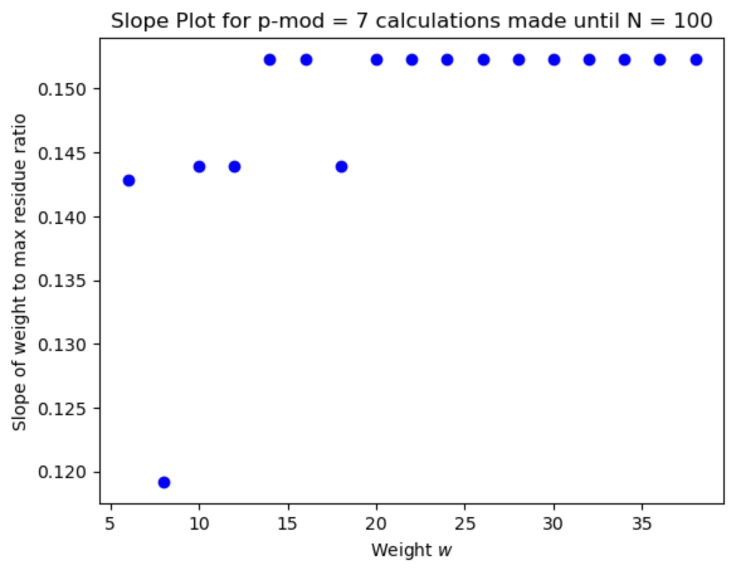
Slope Plot $p = 3$



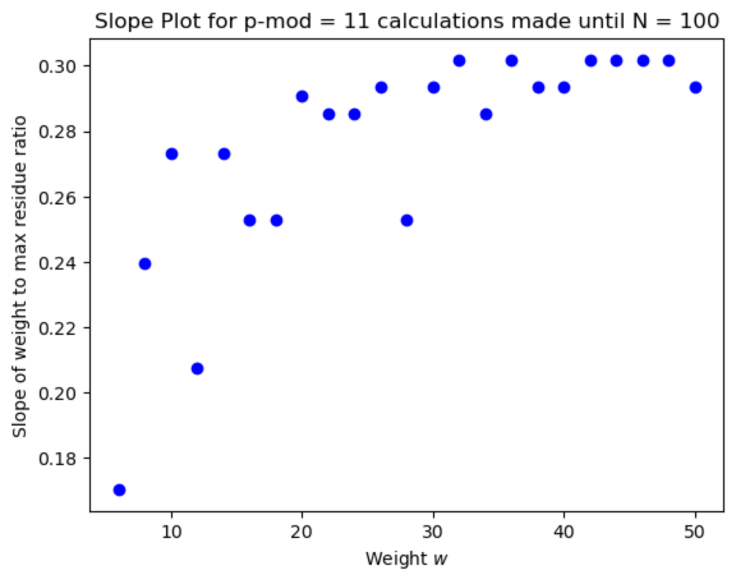
Slope Plot $p = 5$



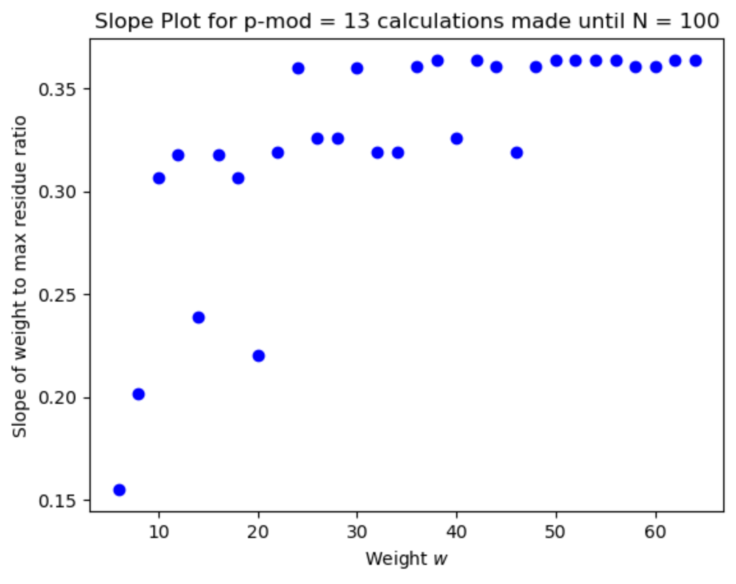
Slope Plot $p = 7$



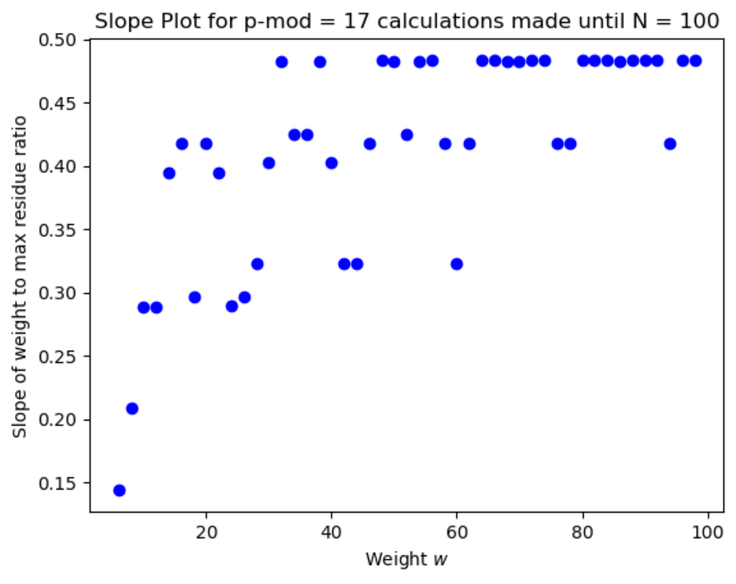
Slope Plot $p = 11$

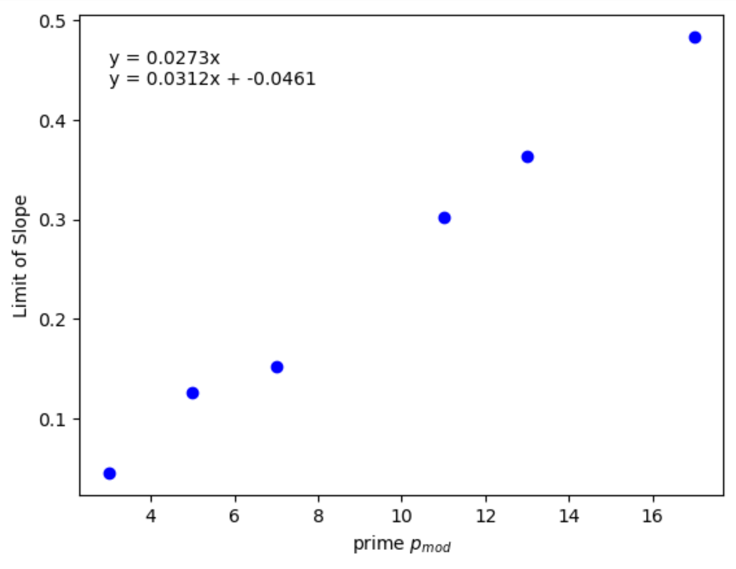


Slope Plot $p = 13$



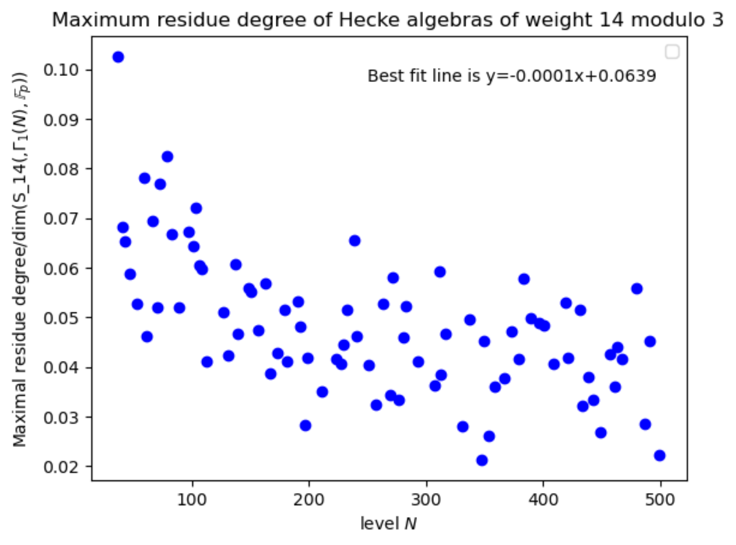
Slope Plot $p = 17$





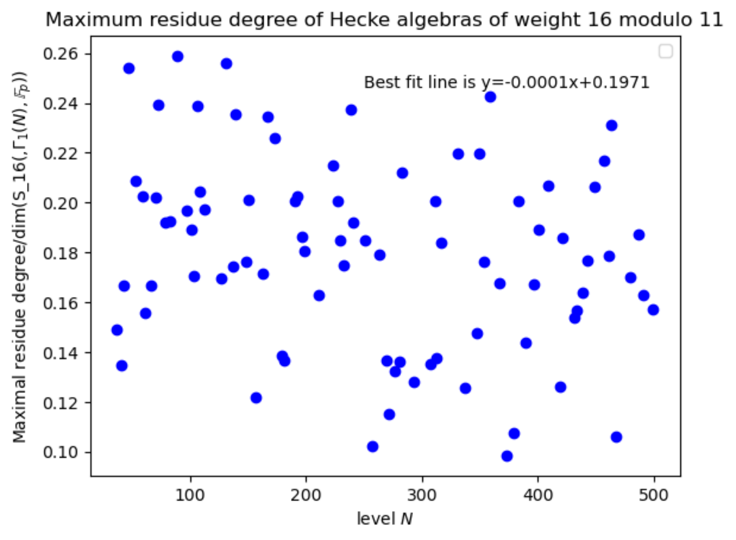
y = 0.0273x
y = 0.0312x + -0.0461
0.5
0.4
0.3
0.2
0.1
Limit of Slope
4
6
8
10
12
14
16
prime p_{mod}

Flats



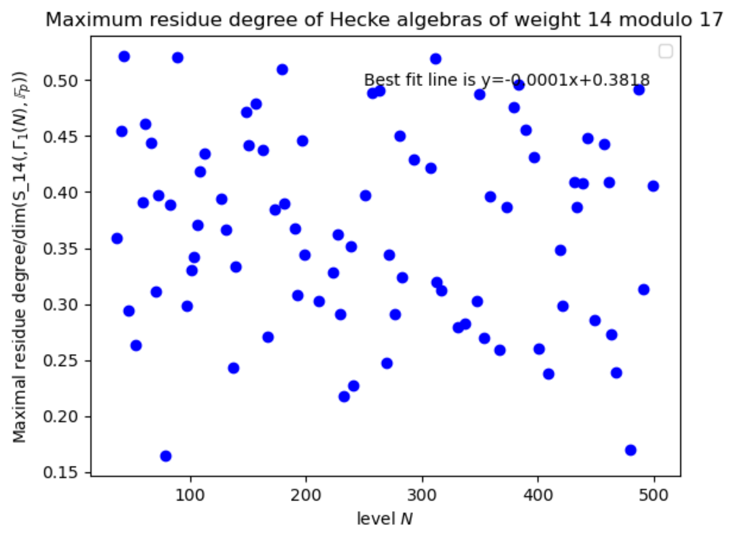
Maximum residue degree of Hecke algebras of weight 14 modulo 3
Best fit line is y=-0.0001x+0.0639
Maximal residue degree/dim(S_14(,Γ_1(N), F_p))
level N
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
100
200
300
400
500

Flats



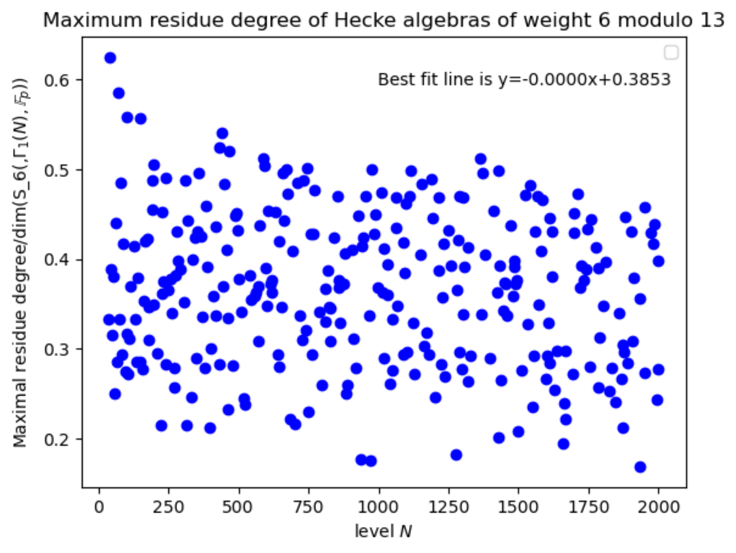
Maximum residue degree of Hecke algebras of weight 16 modulo 11
Best fit line is y=-0.0001x+0.1971
Maximal residue degree/dim(S_16(,Γ1(N), Fp))
level N
0.10
0.12
0.14
0.16
0.18
0.20
0.22
0.24
0.26
100
200
300
400
500

Flats

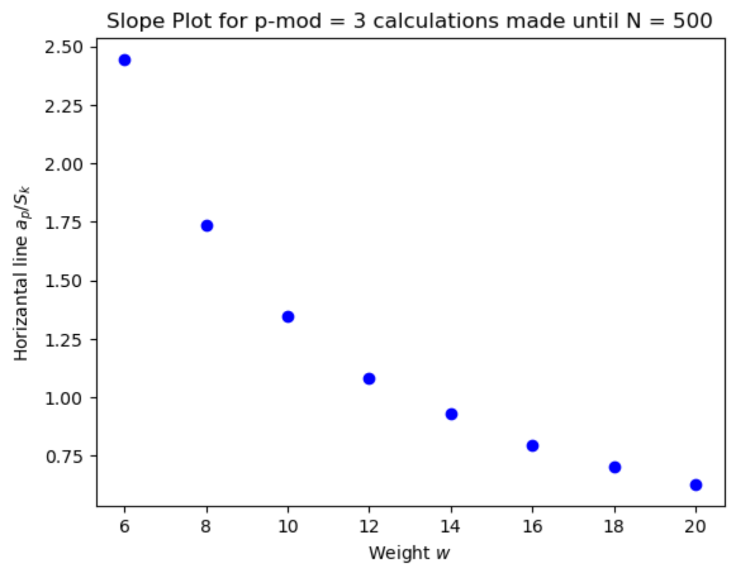


Maximum residue degree of Hecke algebras of weight 14 modulo 17
Best fit line is y=-0.0001x+0.3818
Maximal residue degree/dim(S_14(,Γ_1(N), F_p))
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
100
200
300
400
500
level N

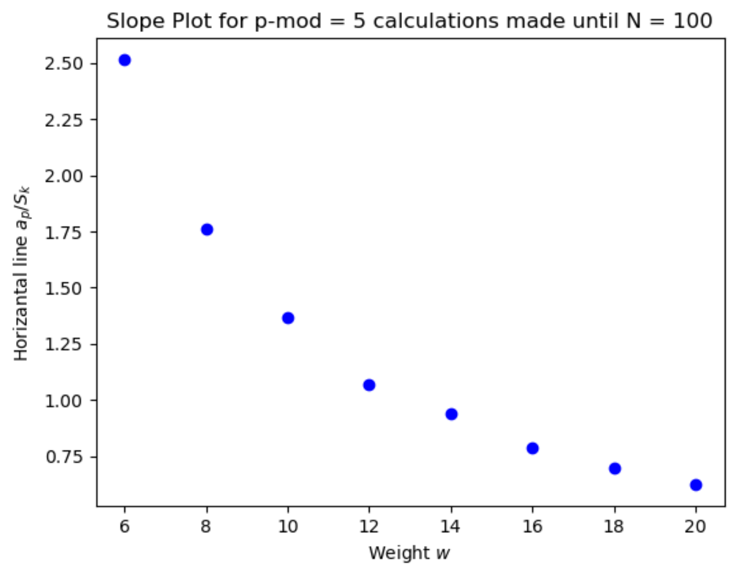
Flats

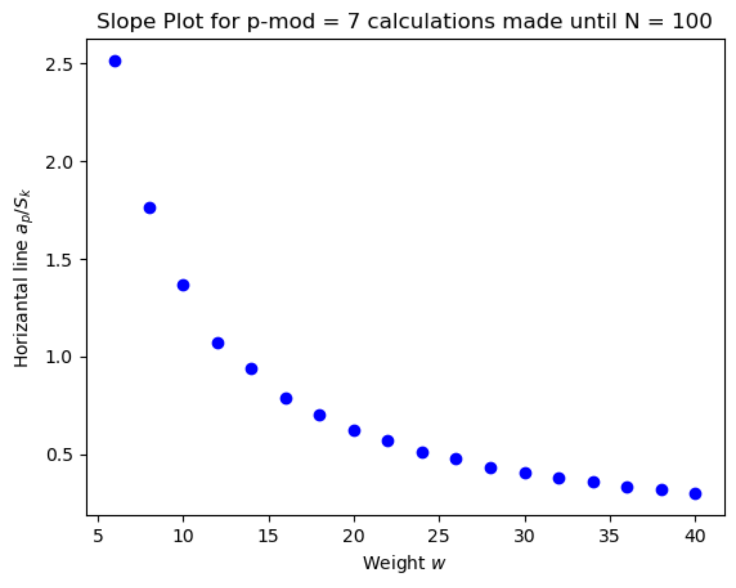


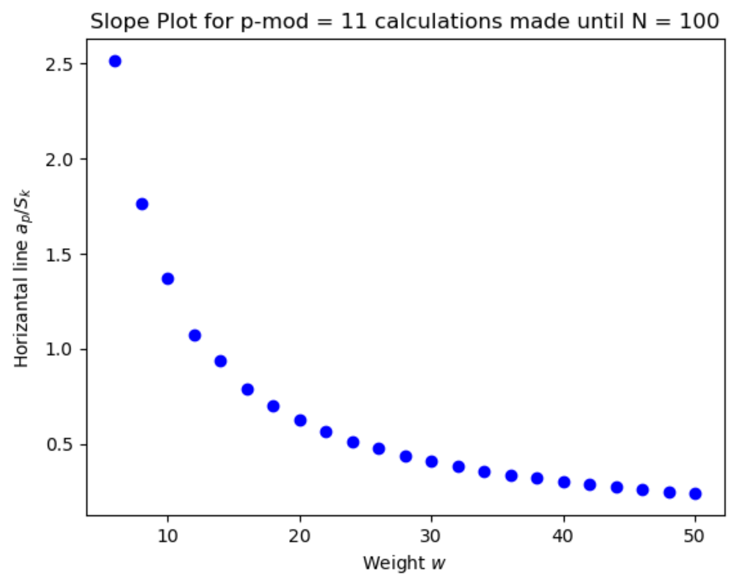
Maximum residue degree of Hecke algebras of weight 6 modulo 13
Best fit line is y=-0.0000x+0.3853
Maximal residue degree/dim(S_6(,$\Gamma_1(N)$, $\mathbb{F}_p$))
0.6
0.5
0.4
0.3
0.2
0
250
500
750
1000
1250
1500
1750
2000
level N

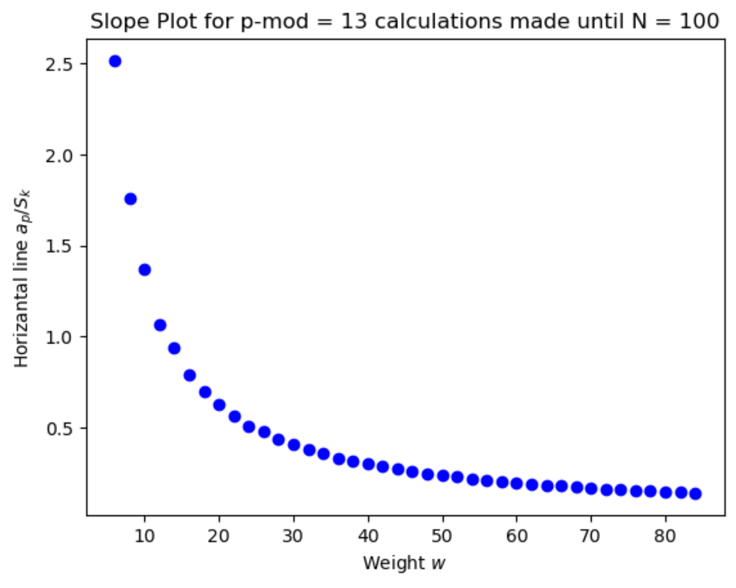


Slope Plot for p-mod = 3 calculations made until N = 500
Horizantal line a_p/S_k
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
6
8
10
12
14
16
18
20
Weight w

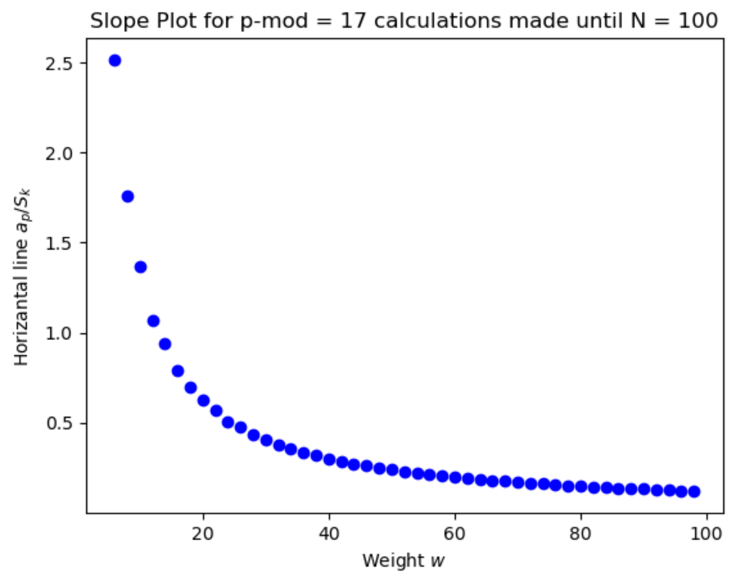

Slope Plot for p-mod = 5 calculations made until N = 100
Horizantal line a_p/S_k
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
6
8
10
12
14
16
18
20
Weight w

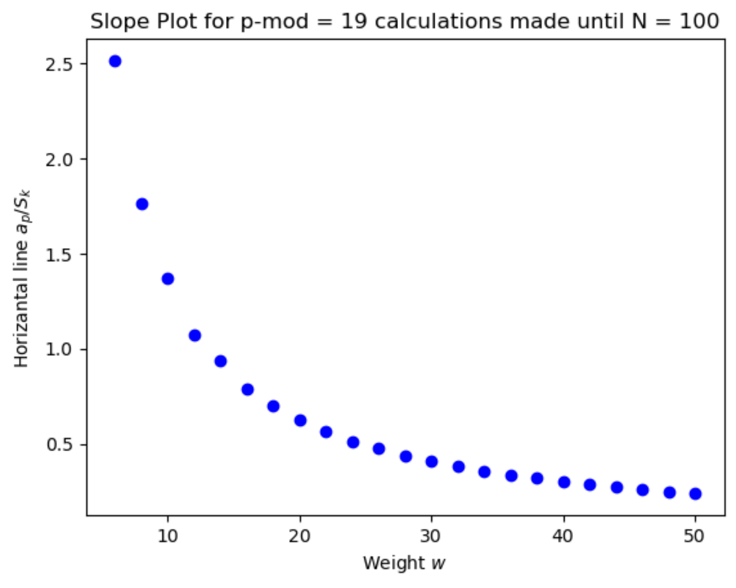

Slope Plot for p-mod = 7 calculations made until N = 100
Horizantal line a_p/S_k
Weight w
2.5
2.0
1.5
1.0
0.5
5
10
15
20
25
30
35
40


Slope Plot for p-mod = 11 calculations made until N = 100
Horizantal line a_p/S_k
Weight w
2.5
2.0
1.5
1.0
0.5
10
20
30
40
50



Slope Plot for p-mod = 13 calculations made until N = 100
Horizantal line a_p/S_k
Weight w
2.5
2.0
1.5
1.0
0.5
10
20
30
40
50
60
70
80


Slope Plot for p-mod = 17 calculations made until N = 100
Horizantal line a_p/S_k
Weight w
2.5
2.0
1.5
1.0
0.5
20
40
60
80
100



Slope Plot for p-mod = 19 calculations made until N = 100
Horizantal line a_p/S_k
Weight w
2.5
2.0
1.5
1.0
0.5
10
20
30
40
50

Back to the Heuristic

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Answer: Yes!

$$\lim_{N\to\infty} a_p(N)/dim(S_k(N;\mathbb{C}) \approx h(p_{mod}, w) \sim 13/w$$

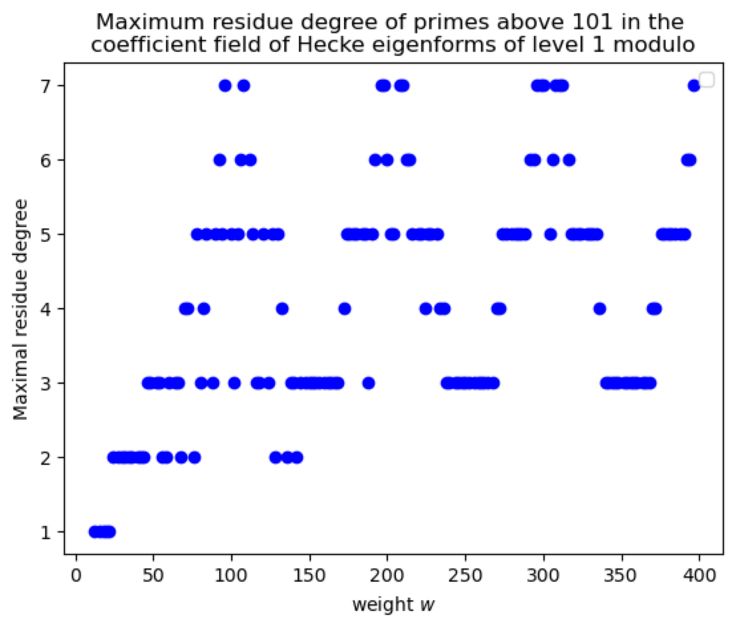
Back to the Heuristic

Question: Is the maximal residue degree, a_p, of primes above p in $\mathbb{Q}_f$ related to b_n, the average maximum length of a cycle in a permutation of $\mathcal{S}_n$?
Answer: Yes!

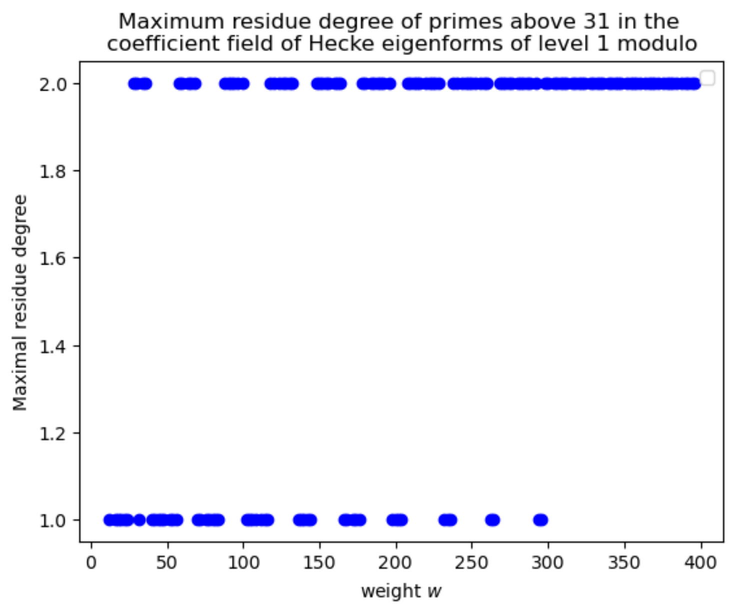
$$\lim_{N\to\infty} a_p(N)/dim(S_k(N;\mathbb{C}) \approx h(p_{mod}, w) \sim 13/w$$

Questions?

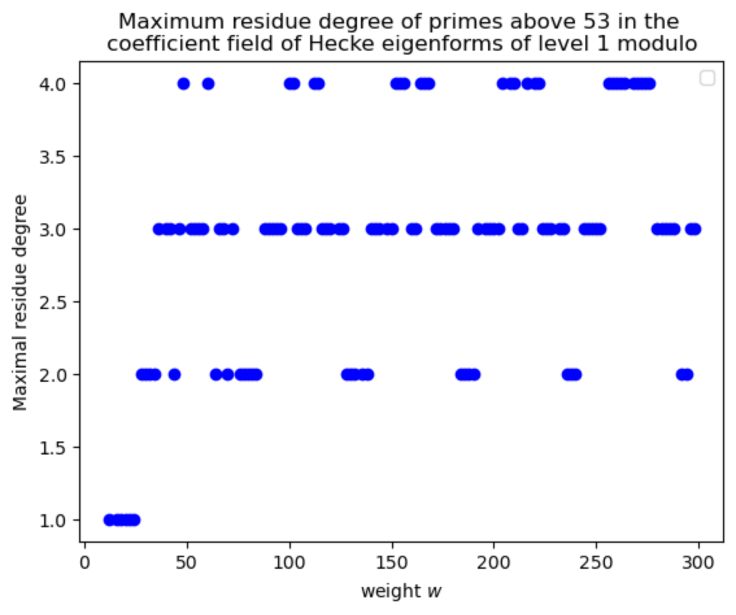
Plots: Fixed Level $N = 1$



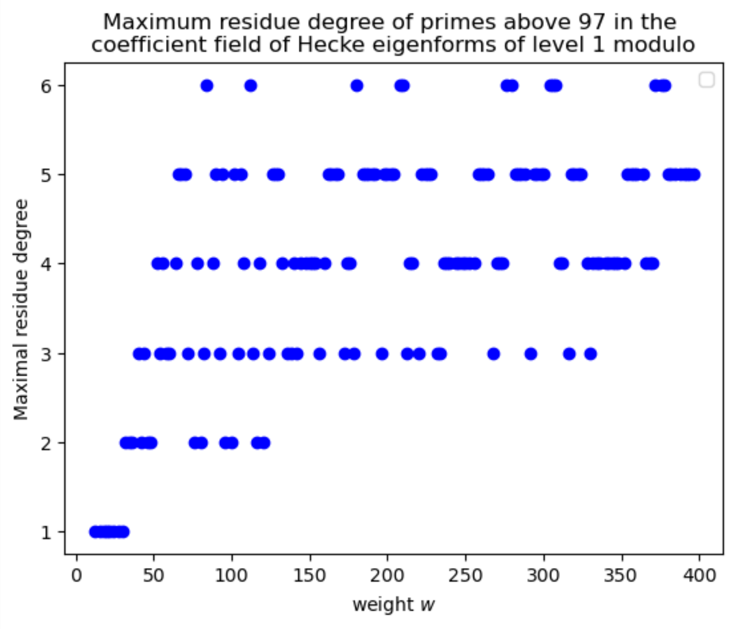
Plots: Fixed Level $N = 1$



Plots: Fixed Level $N = 1$



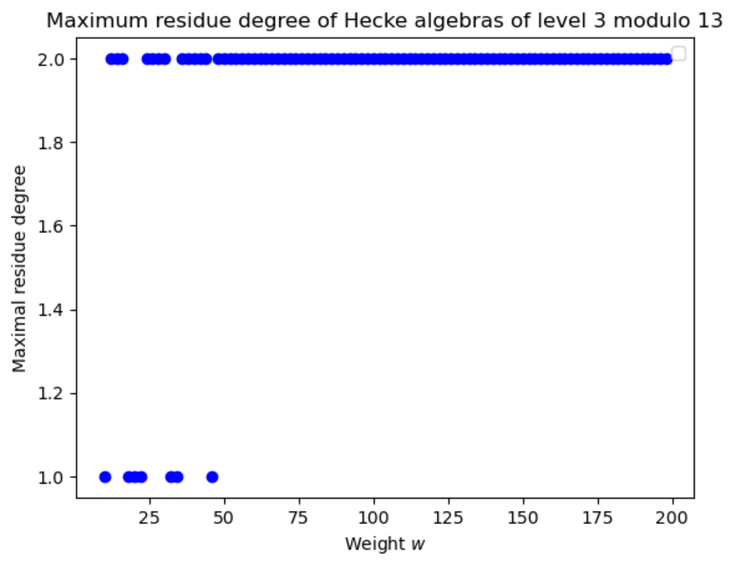
Plots: Fixed Level $N = 1$



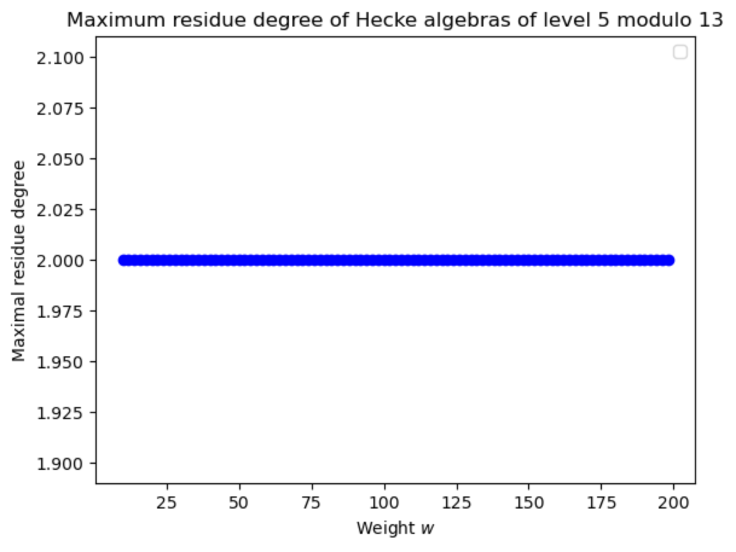
Jochnowitz

If we know all eigenforms of level 1 and weight $\leq p+1$, then we essentially get all the eigenforms over $\mathbb{F}_p$ in all weights by multiplying those of low weights by A_p, where $A_p = 1$ is a modular form of weight $p-1$ and level 1 over $\mathbb{F}_p$.

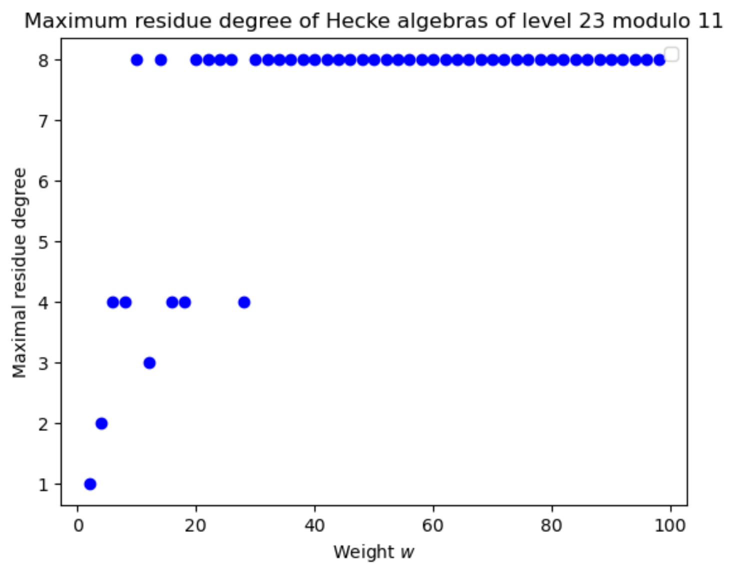
Fixed Level: $N \geq 1$



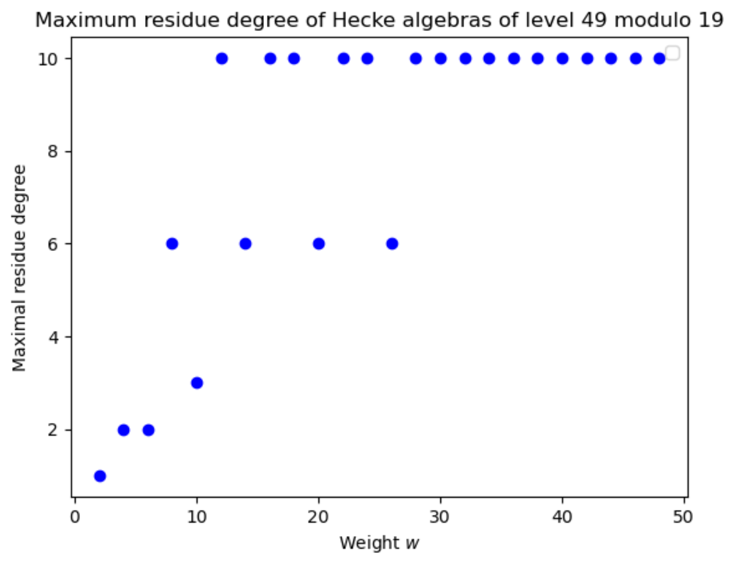
Fixed Level: $N \geq 1$



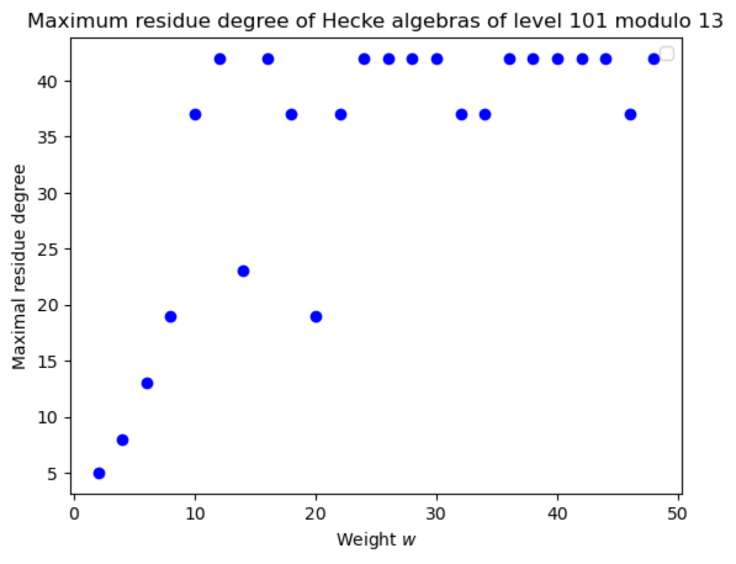
Fixed Level: $N \geq 1$



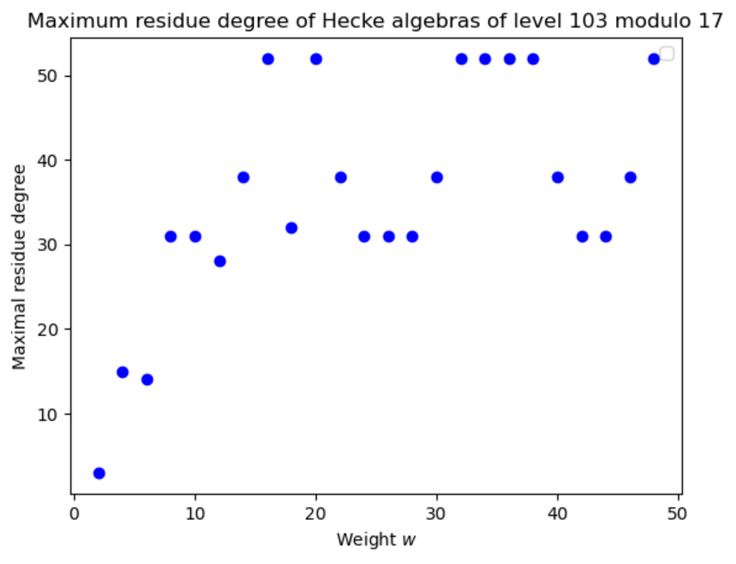
Fixed Level: $N \geq 1$



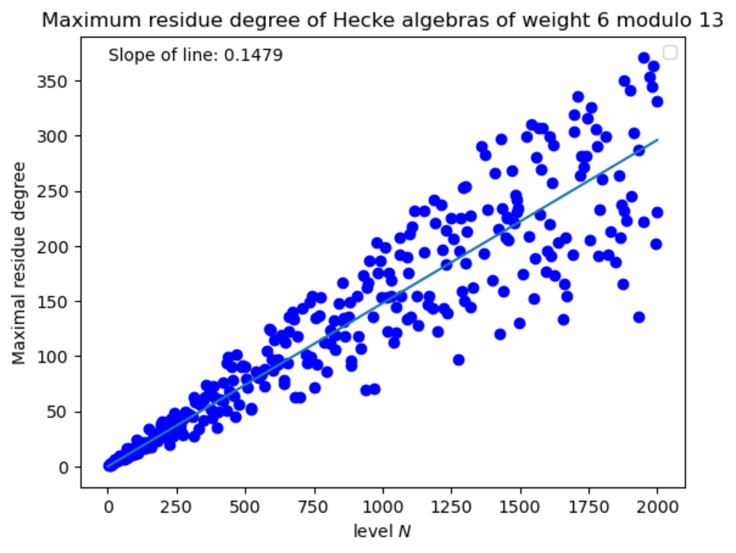
Fixed Level: $N \geq 1$



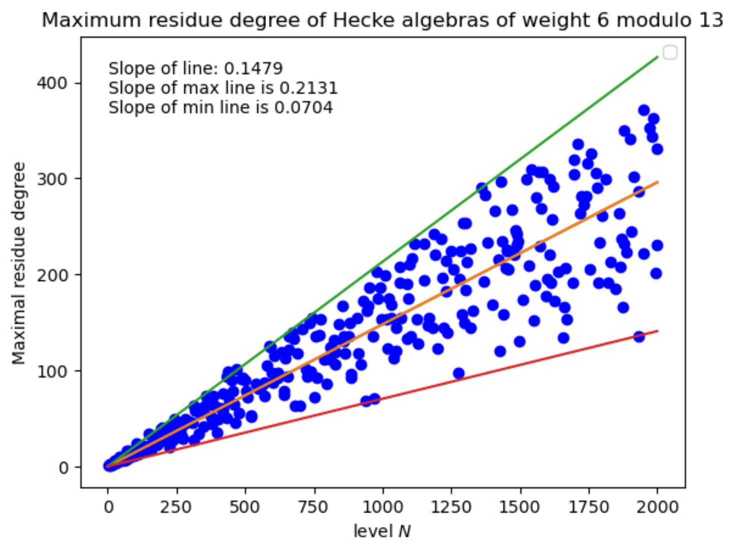
Fixed Level: $N \geq 1$



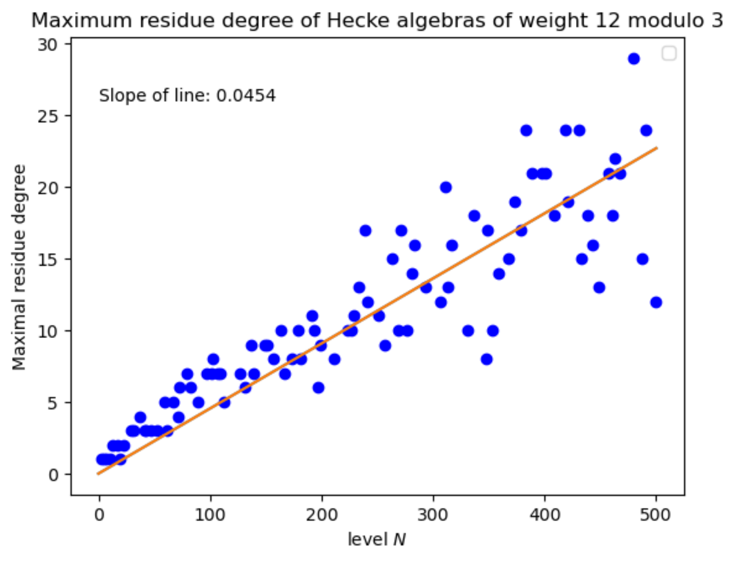
Fixed Weight



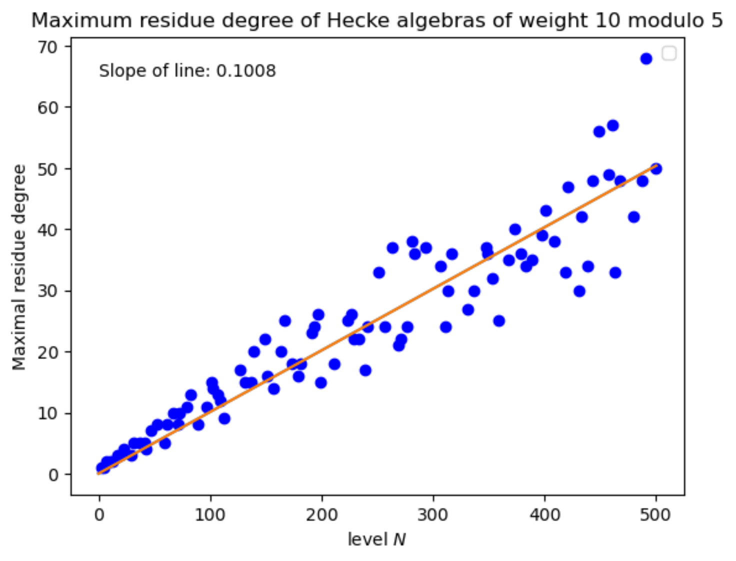
Fixed Weight



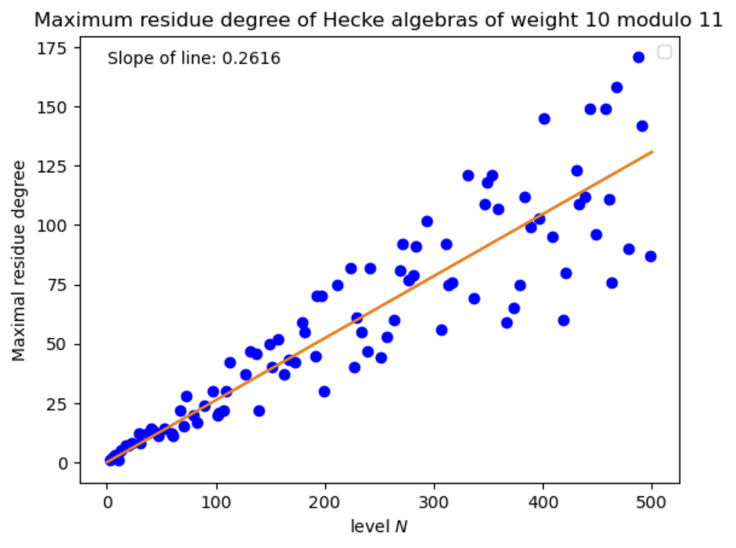
Fixed Weight



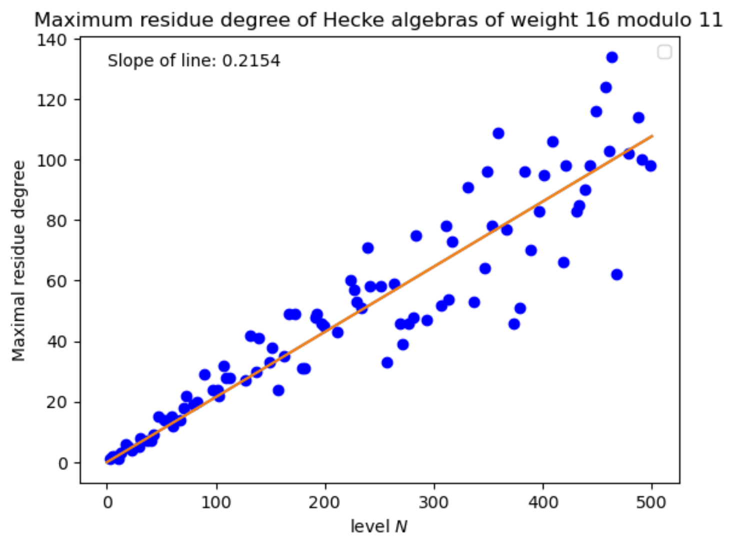
Fixed Weight



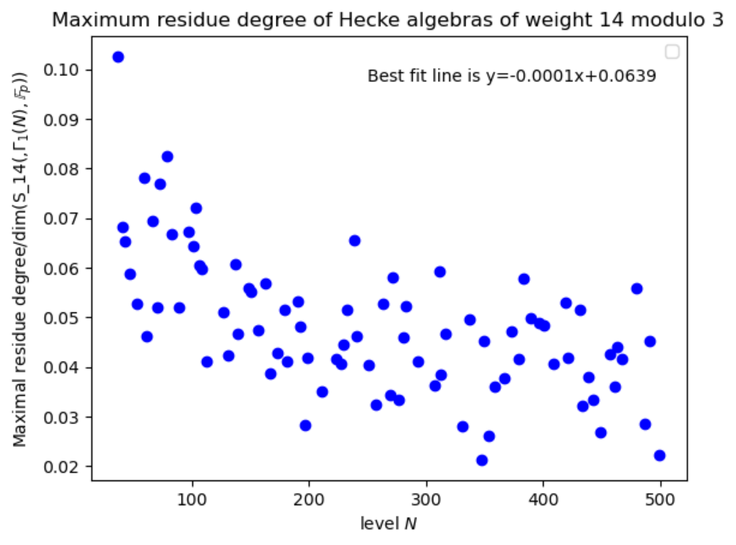
Fixed Weight



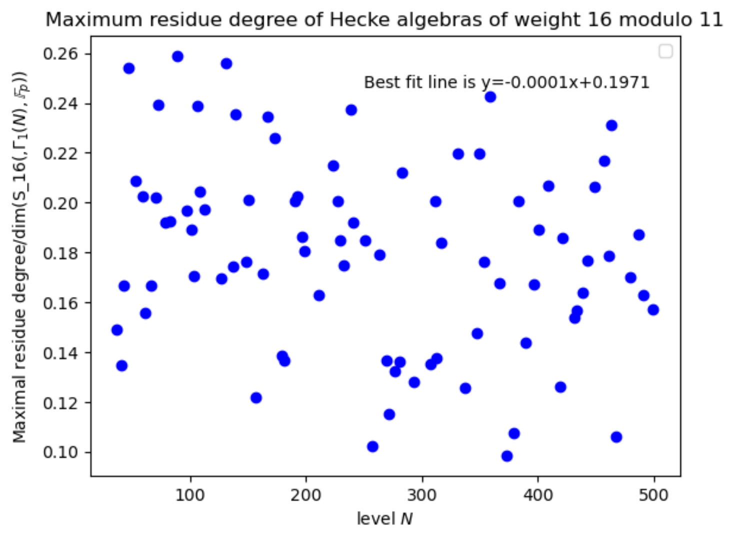
Fixed Weight



Flats

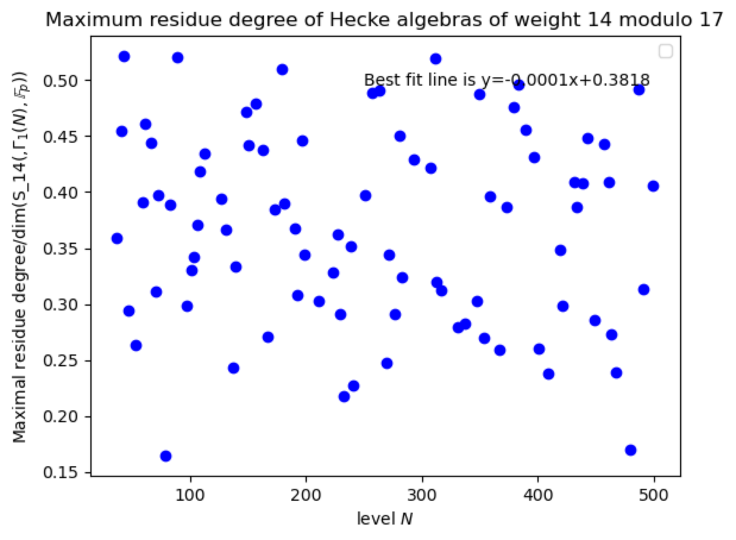

Maximum residue degree of Hecke algebras of weight 14 modulo 3
Best fit line is y=-0.0001x+0.0639
Maximal residue degree/dim(S_14(,Γ_1(N), $\mathbb{F}_p$))
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
100
200
300
400
500
level N

Flats



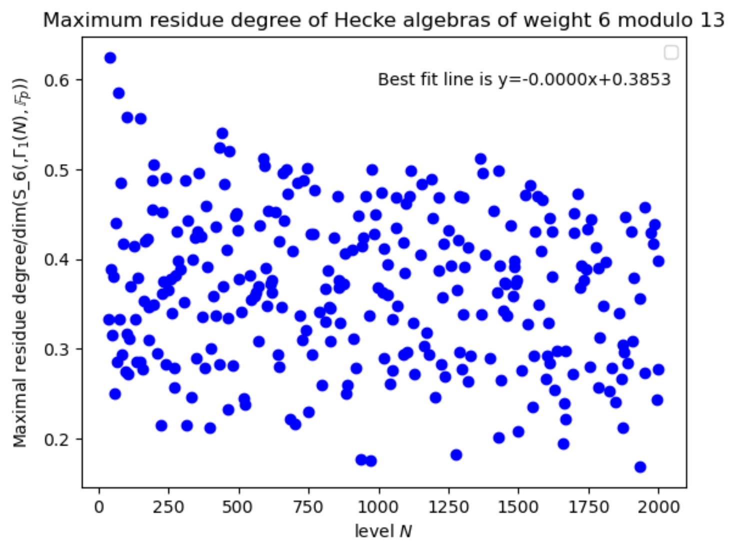
Maximum residue degree of Hecke algebras of weight 16 modulo 11
Best fit line is y=-0.0001x+0.1971
Maximal residue degree/dim(S_16(,Γ1(N), Fp))
level N
0.26
0.24
0.22
0.20
0.18
0.16
0.14
0.12
0.10
100
200
300
400
500

Flats

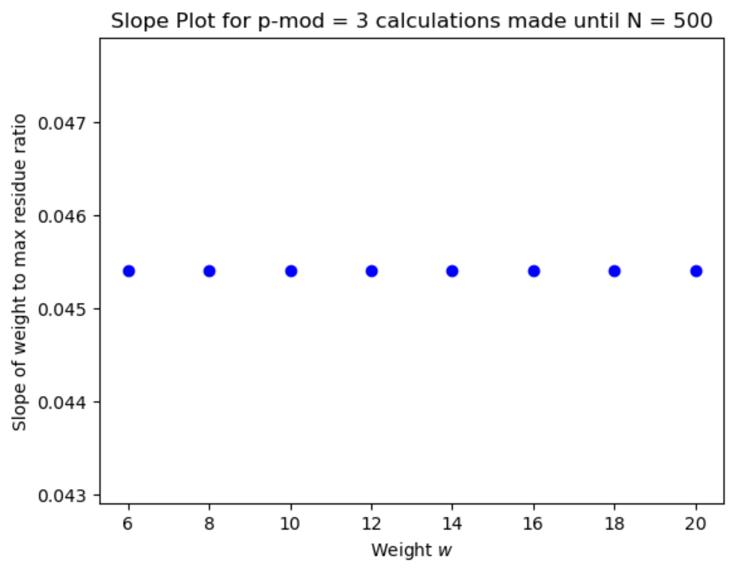


Maximum residue degree of Hecke algebras of weight 14 modulo 17
Best fit line is y=-0.0001x+0.3818
Maximal residue degree/dim(S_14(,Γ_1(N), F_p))
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
100
200
300
400
500
level N

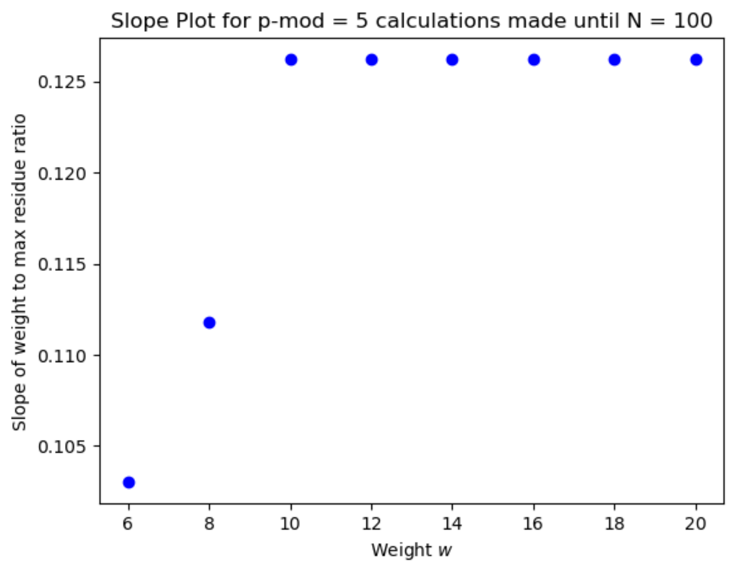
Flats


Maximum residue degree of Hecke algebras of weight 6 modulo 13
Best fit line is y=-0.0000x+0.3853
Maximal residue degree/dim(S_6(,Γ_1(N), F_p))
level N
0.2
0.3
0.4
0.5
0.6
0
250
500
750
1000
1250
1500
1750
2000

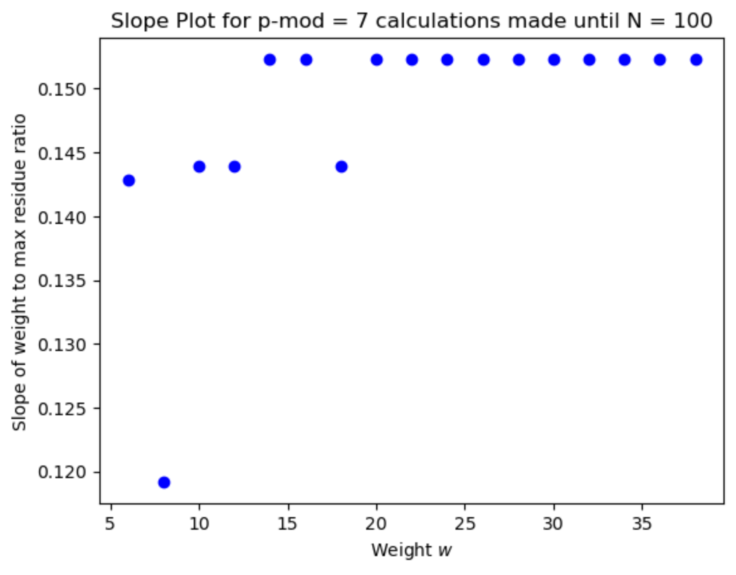
Slope Plot $p = 3$



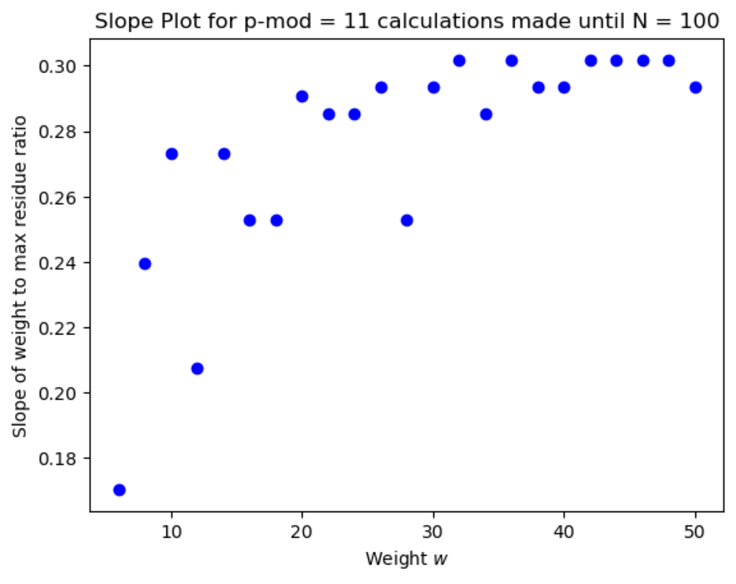
Slope Plot $p = 5$



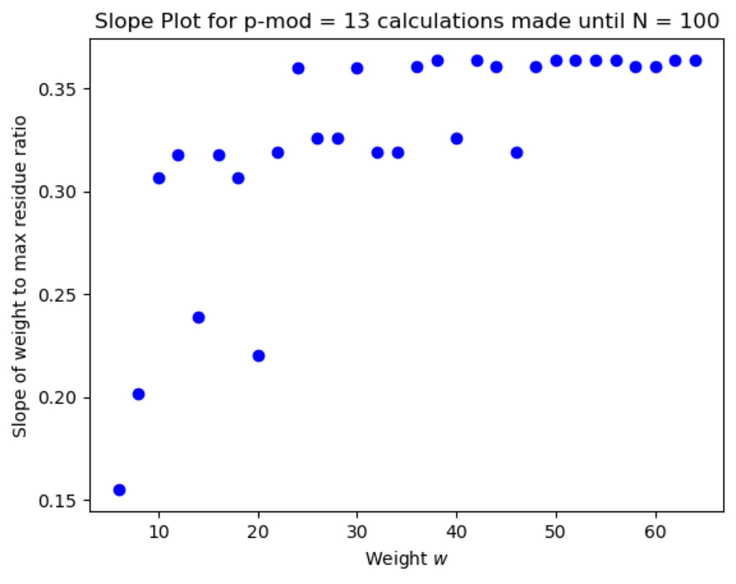
Slope Plot $p = 7$



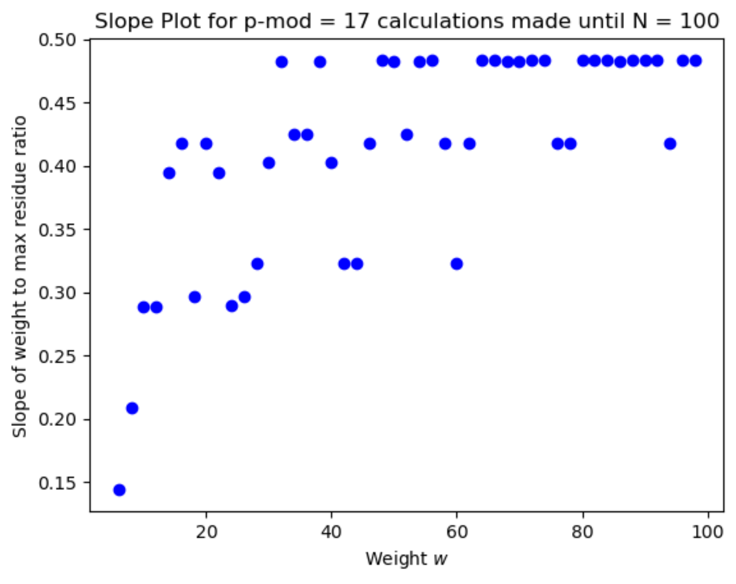
Slope Plot $p = 11$

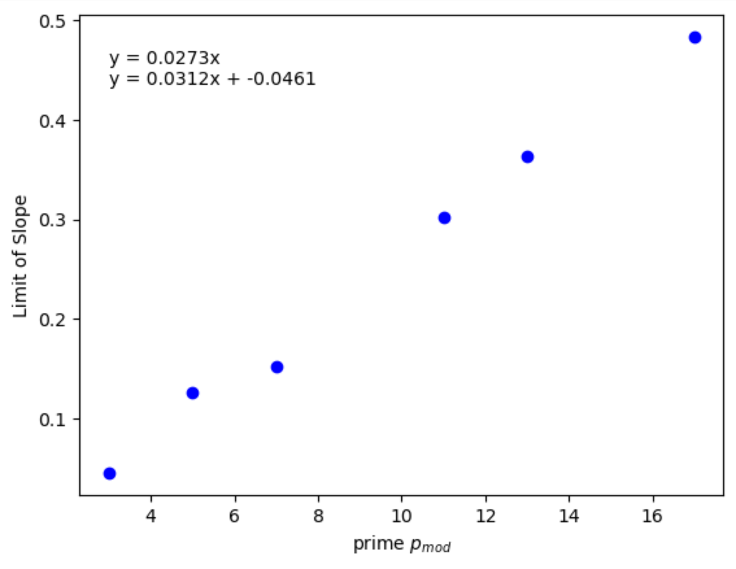


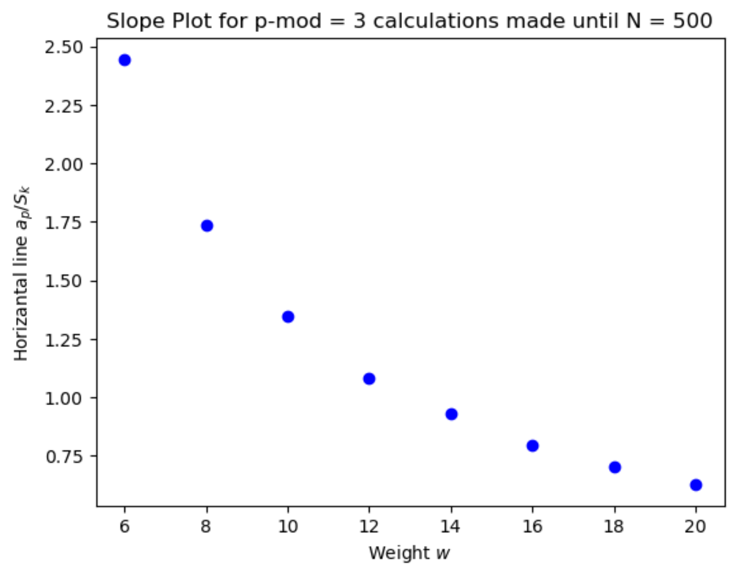
Slope Plot $p = 13$

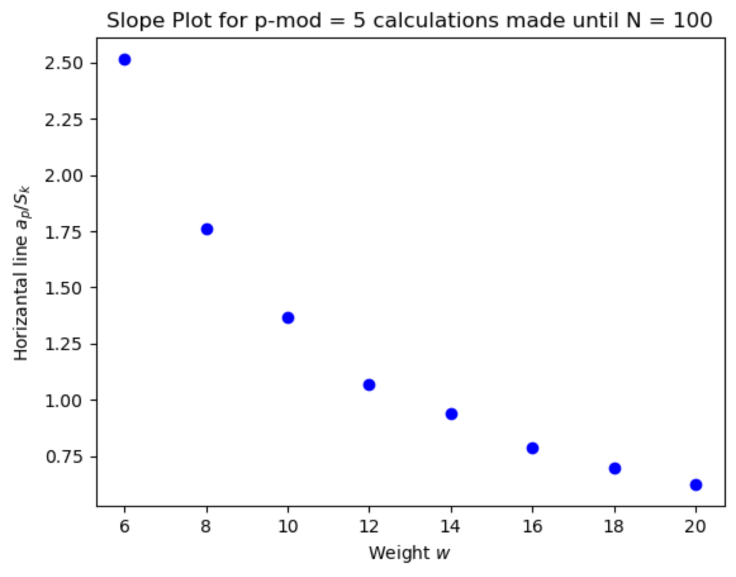


Slope Plot $p = 17$

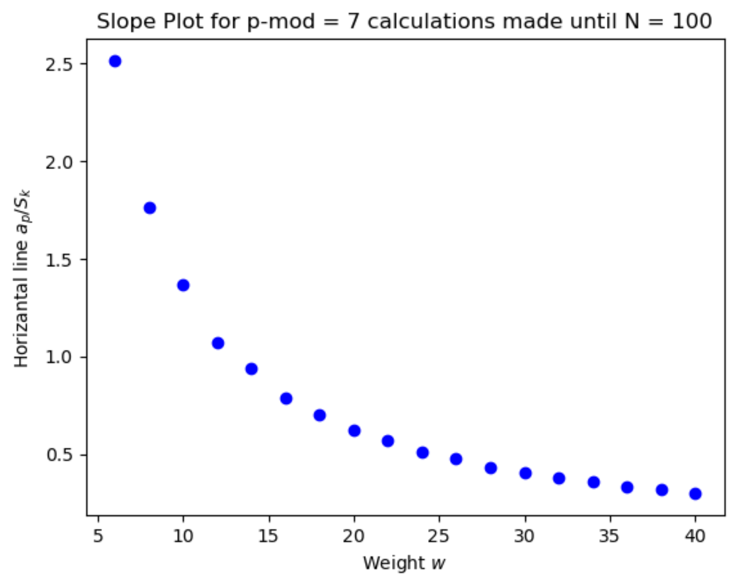



y = 0.0273x
y = 0.0312x + -0.0461
0.5
0.4
0.3
0.2
0.1
Limit of Slope
4
6
8
10
12
14
16
prime p_{mod}

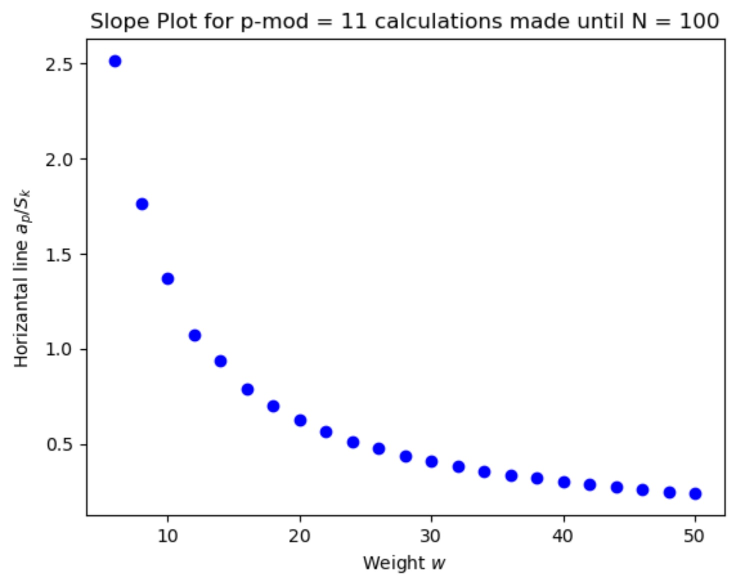

Slope Plot for p-mod = 3 calculations made until N = 500
Horizantal line a_p/S_k
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
6
8
10
12
14
16
18
20
Weight w

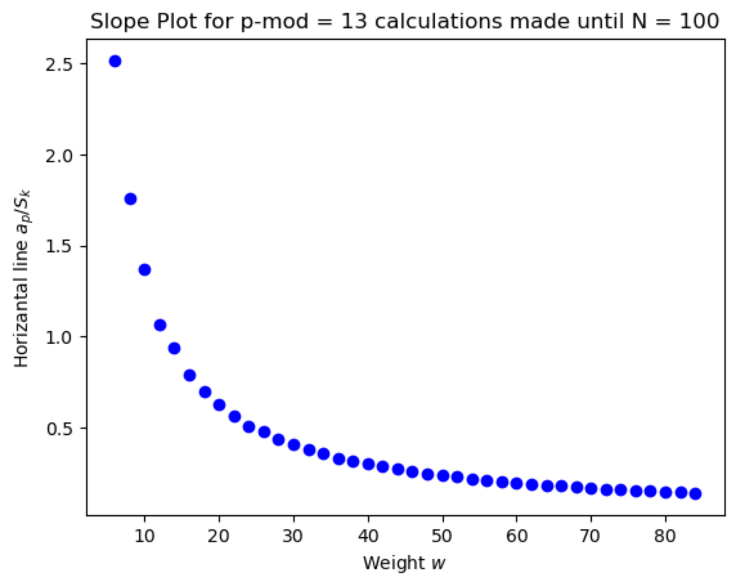


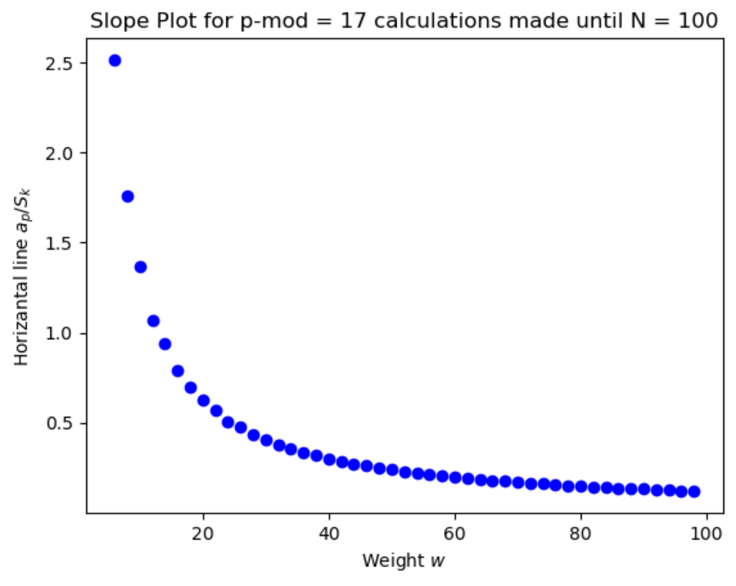
Slope Plot for p-mod = 5 calculations made until N = 100
Horizantal line a_p/S_k
Weight w
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
6
8
10
12
14
16
18
20



Slope Plot for p-mod = 7 calculations made until N = 100
Horizantal line a_p/S_k
Weight w
2.5
2.0
1.5
1.0
0.5
5
10
15
20
25
30
35
40


Slope Plot for p-mod = 11 calculations made until N = 100
Horizantal line a_p/S_k
2.5
2.0
1.5
1.0
0.5
10
20
30
40
50
Weight w


Slope Plot for p-mod = 13 calculations made until N = 100
Horizantal line a_p/S_k
Weight w
2.5
2.0
1.5
1.0
0.5
10
20
30
40
50
60
70
80


Slope Plot for p-mod = 17 calculations made until N = 100
Horizantal line a_p/S_k
Weight w
2.5
2.0
1.5
1.0
0.5
20
40
60
80
100